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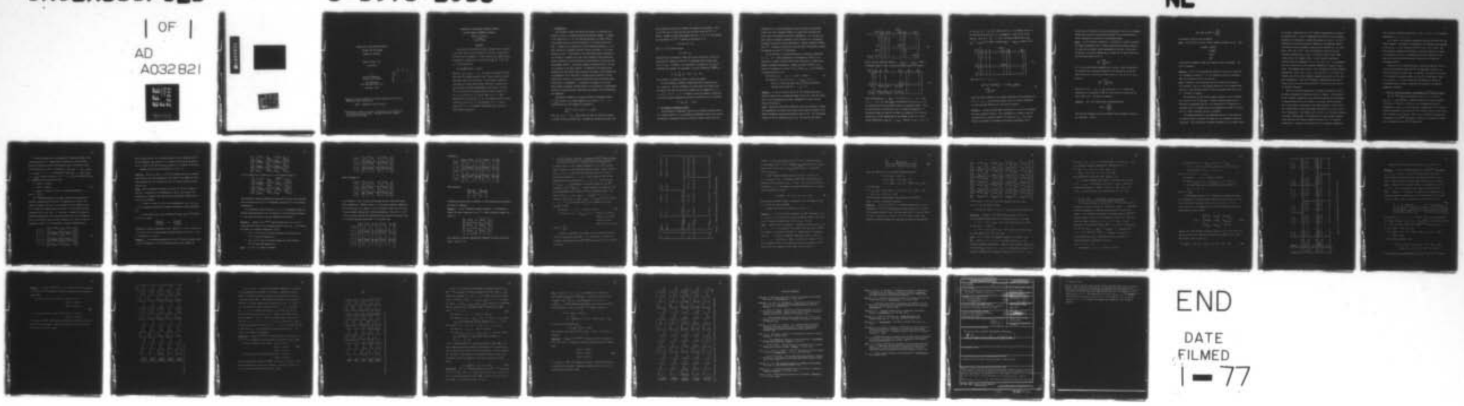

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RESOLUTION IV FRACTIONAL FACTORIAL DESIGNS FOR THE GENERAL ASYMMETRIC FACTORIAL

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RESOLUTION IV FRACTIONAL FACTORIAL DESIGNS

FOR THE GENERAL ASYMMETRIC FACTORIAL

Donald A. Anderson
and
Ann M. Thomas

ABSTRACT

Resolution IV fractional factorial designs permit estimates of all main effects in the presence of two factor interactions which may not be estimable. A lower bound on the number of runs required for a resolution IV design in the $\prod_{i=1}^{k} s_i^{n_i} \times s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k}$ factorial is

$$N \geq s_k[\sum_{i=1}^{k}(s_i-1)n_i - (s_k-2)]$$

where $s_k \geq s_{k-1}, s_{k-2}, \ldots, s_1$. No series of designs are known which meet this bound except for the 2^n series and the trivial case where there are two factors. In this paper a method of construction is given which yields resolution IV designs near the theoretical (perhaps unattainable) lower bound. For the $2^m \times 3^n$ factorial, the designs exceed the lower bound by six if $n \geq 3$, by four if $n = 2$, and by three if $n = 1$. More generally for the $s_1^n \times s_2^n$, $s_1 < s_2$, the designs exceed the lower bound by $s_2(s_2-1)$ if $n \geq 3$, by $s_2^2-3s_2+2s_1$ if $n = 2$, and by $s_2(s_1-1)$ if $n = 1$. In general the designs never exceed the lower bound by $s_k(s_k-1)$.

1. Introduction

Box and Hunter (1961) introduced the concept of resolution of a design as one way to classify fractional factorial designs. A design is of resolution 2r + 1 if all effects involving r or fewer factors are estimable when all effects involving r + 1 or more factors are zero. A design is of resolution 2r if all effects involving r - 1 or fewer factors are estimable when all effects involving r + 1 or more factors are zero. Thus designs of odd resolution permit estimation of all effects not assumed to be zero, while designs of even resolution permit estimation of certain effects in the presence of other non-zero, nonestimable effects. In practice designs of resolutions III, IV, and V are perhaps of the most interest. A resolution III design allows estimation of main effects when two-factor and higher order interactions are negligible, and a resolution V design allows estimation of main effects and two-factor interactions when three-factor and higher order interactions are negligible. A resolution IV design, on the other hand, permits estimation of main effects in the presence of nonestimable two-factor interactions when three-factor and higher order interactions are negligible.

The general asymmetric, or mixed, factorial experiment involves $n = \sum_{i=1}^{k} n_i$ factors, n_i of which appear at s_i levels, $i = 1, \ldots, k$. Usual notation for the asymmetric experiment is

$$\prod_{i=1}^{k} s_i^{n_i} = s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k} ,$$

where $s_1 < s_2 < \ldots < s_k$. (When there are only two different numbers of levels we will write $s_1^m \times s_2^n$.) Statistical literature provides very few incomplete resolution IV designs for asymmetric experiments. Most of the work done in this area has been directed toward the $2^m \times 3^n$ series. Margolin (1969a) established that for $n > 0$, $m \geq 0$, the minimum run requirement for $2^m \times 3^n$ designs of resolution IV is

$$N \geq 3(m + 2n - 1) .$$

When $n = 1$, this bound becomes

$$N \geq 3(m + 1) ,$$

and Anderson and Srivastava (1969, 1972) have constructed a series of resolution IV designs for the $2^m \times 3^n$ experiment which require only $4(m + 1)$ runs. In general, for resolution IV designs for the $\prod_{i=1}^{k} s_i^{n_i}$ experiment the minimum run requirement is given by Margolin (1969a) as

$$N \geq s_k[\sum_{i=1}^{k} (s_i - 1)n_i - (s_k - 2)] ,$$

but no series of designs is known to attain this bound.

In the following a treatment combination will be denoted as an $n \times 1$ vector $\underline{t}$. The ith coordinate of $\underline{t}$ will denote the level of the ith factor, and these levels will be denoted by the symbols 0, 1, 2, ..., s_i-1, $i = 1, 2, \ldots, n$. A design in N runs is simply a collection of N such treatment combinations and will be denoted as an $n \times N$ matrix

$$T = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_N] .$$

2. The Method of Collapsing Levels

One procedure of replacing a factor at s_2 levels by another factor at s_1 levels, where $s_1 < s_2$, is known as collapsing levels of the factor. The method of collapsing levels as a means of design construction was introduced in a concise mathematical form by Kishen and Srivastava (1959), and also by Addelman (1962) in a paper that developed main effect plans for asymmetric factorial experiments. The technique has been applied frequently, primarily in the construction of orthogonal designs of resolution III and resolution V. However, Margolin (1969b) used 3^{m+n} designs of resolution IV as base plans from which to derive new designs for the $2^m \times 3^n$ experiment.

Let the levels of a factor appearing at s levels be labelled as 0, 1, ..., s-1. When considering the collapsing of levels, it is convenient to define the main effects for each factor in terms of the Helmert orthogonal polynomials. Suppose now that a factor F at s_2 levels is collapsed to a factor F^c at s_1 levels, $s_1 < s_2$, by means of a mapping as shown in (1).

For levels 0, 1, ..., $s_1 - 1$: level $\ell \rightarrow$ level ℓ (1)

For levels $\ell_1 = s_1$, $\ell_2 = s_1 + 1$, ..., $\ell_{s_2-s_1} = s_2 - 1$:

level $\ell_j \rightarrow$ level q_j, where each q_j is one arbitrarily selected level from $\{0, 1, ..., s_1 - 1\}$.

<u>Theorem 1</u>. If the s_1 levels of Factor F^c are obtained from the s_2 levels of Factor F by a collapsing scheme of type (1), each main effect for F^c can be expressed as a linear combination of μ and the main effects for Factor F.

<u>Proof</u>. Since main effects are defined in terms of the Helmert orthogonal polynomials, the proof can be constructed using the columns of the Helmert polynomial system partitioned as shown in (2). The individual columns in (2) are labelled by $\underline{a}_0 = \underline{1}$ for the mean, $\underline{a}_1$ for linear

Level	$\underline{a}_0$	$\underline{a}_1$	$\underline{a}_2$	$\cdots$	$\underline{a}_{s_1-1}$	$\underline{a}_{s_1}$	$\cdots$	$\underline{a}_{s_2-1}$
0	1	1	1	$\cdots$	1	1	$\cdots$	1
1	1	-1	1		1	1		1
2	1	0	-2		1	1		1
$\vdots$	$\vdots$	$\vdots$	0 $\vdots$		$\vdots$	$\vdots$		$\vdots$
s_1-1	1	0	0	$\cdots$	$-(s_1-1)$	1	$\cdots$	1
$\ell_1=s_1$	1	0	0	$\cdots$	0	$-s_1$	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	0 $\vdots$		$\vdots$
$\ell_{s_2-s_1}=s_2-1$	1	0	0	$\cdots$	0	0	$\cdots$	$-(s_2-1)$

(2)

,

effects, $\underline{a}_2$ for quadratic effects, ..., $\underline{a}_{s_1-1}$, ..., $\underline{a}_{s_2-1}$. Under a collapsing scheme of type (1) the partitioned system (2) becomes

Level	$\underline{a}_0^c$	$\underline{a}_1^c$	$\underline{a}_2^c$	$\cdots$	$\underline{a}_{s_1-1}^c$	$\underline{a}_{s_1}^c$	$\cdots$	$\underline{a}_{s_2-1}^c$
0	1	1	1	$\cdots$	1	1	$\cdots$	1
1	1	-1	1		1	1		1
2	1	0	-2		$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$	$\vdots$	0 $\vdots$		$\vdots$	$\vdots$		$\vdots$
s_1-1	1	0	0	$\cdots$	$-(s_1-1)$	1	$\cdots$	1
q_1	1	$a_1(q_1)$	$a_2(q_1)$	$\cdots$	$a_{s_1-1}(q_1)$	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$q_{s_2-s_1}$	1	$a_1(q_{s_2-s_1})$	$a_2(q_{s_2-s_1})$	$\cdots$	$a_{s_1-1}(q_{s_2-s_1})$	1	$\cdots$	1

(3)

with columns $\underline{a}_0^c$, $\underline{a}_1^c$, ..., $\underline{a}_{s_2-1}^c$. In (3) for $k = 1, \ldots, s_1 - 1$ and $j = 1, \ldots, s_2 - s_1$, each $a_k(q_j)$ is determined by (2) as 0, 1, or $-k$ in accordance with the particular choice of collapsing scheme. Since all main effects are defined in terms of Helmert polynomials, it suffices to show that each of the first s_1 columns of (3) can be expressed as a linear combination of the columns of (2), i.e., as a linear combination of $\underline{a}_0$, $\underline{a}_1$, ..., $\underline{a}_{s-1}$. Now for $k = s_1, \ldots, s_2 - 1$

in (2), $\underline{a}_k = [\underline{1} \mid -s_k \mid \underline{0}]'$, where $\underline{1}$ is a $(k-1)$-component vector of all ones and $\underline{0}$ is an $[(s_2-1)-k]$-component vector of all zeros. Thus $\underline{a}_k - \underline{a}_0 = [\underline{0} \mid -s_k - 1 \mid -\underline{1}]'$, so linear combinations of $\underline{a}_0$, $\underline{a}_{s_1}, \ldots, \underline{a}_{s_2}$ can be taken to produce $\underline{a}^*_{s_1}, \ldots, \underline{a}^*_{s_2-1}$ as shown in (4).

Level	Effect							
	$\underline{a}_0$	$\underline{a}_1$	$\underline{a}_2$	$\cdots$	$\underline{a}_{s_1-1}$	$\underline{a}^*_{s_1}$	$\cdots$	$\underline{a}^*_{s_2-1}$
0	1	1	1	$\cdots$	1	0	$\cdots$	0
1	1	-1	1		1	0		0
2	1	0	-2		1	0		0
3	1	0	0		1	0		0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
s_1-1	1	0	0		$-(s_1-1)$	0	$\cdots$	0
s_1	1	0	0	$\cdots$	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	0		0
						$\vdots$		$\vdots$
s_2-1	1	0	0	$\cdots$	0	0	$\cdots$	1

(4)

Then for $k = 1, \ldots, s_1 - 1$

$$\begin{aligned} \underline{a}^c_k &= \underline{a}_k + a_k(q_1)\underline{a}^*_{s_1} + \cdots + a_k(q_{s_2-s_1})\underline{a}^*_{s_2-1} \\ &= \sum_{m=0}^{s_2-1} c_m \underline{a}_m . \end{aligned} \tag{5}$$

Since the linear combination of columns specified by (5) determines for $k = 1, \ldots, s_1 - 1$ the kth order effect of F^c as a linear combination of μ and the main effects of F, the proof is complete.

<u>Theorem 2</u>. Consider an experiment consisting of n factors, and suppose that the s_2 levels of Factor F are collapsed by a scheme of type (1) to produce the s_1 levels of Factor F^c, where $s_1 < s_2$. If all main effects are defined in accordance with Helmert polynomials and if

interactions are defined by the product definition, then in the collapsed design each two-factor interaction involving F^c can be expressed as a linear combination of effects in the original design.

Proof. As in the proof of Theorem 1, it suffices to consider a system of Helmert polynomials since a linear combination involving columns of (2) specifies the needed linear combination of effects. By the proof of Theorem 1 the linear combination of columns of (2) which defines the kth order main effect of F^c is

$$\underline{a}_k^c = \sum_{m=0}^{s_2-1} c_m \underline{a}_m .$$

Let G be any other factor appearing at s_2 levels. Since interactions are specified by the product definition, the interaction of the kth order effect of F^c with the rth order effect of G is, for any k and r, given by

$$\underline{a}_{kr}^c = \sum_{m=0}^{s_2-1} c_m \underline{a}_{mr}$$

where for $m = 0, \ldots, s_2 - 1$, $\underline{a}_{mr}$ is the $(s_2 - 1) \times 1$ vector that determines the interaction of the mth order effect of F with the rth order effect of G. The proof is complete.

Theorem 3. Let T be a design with corresponding model

$$E(\underline{Y}) = X \begin{bmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \end{bmatrix}$$

such that the elements of $\underline{\beta}_1$ are estimable and the elements of $\underline{\beta}_2$ are not estimable. Then if

$$\underline{\beta}_1^* = H_1\underline{\beta}_1 \text{ and } \underline{\beta}_2^* = H_2 \begin{bmatrix} \underline{\beta}_1 \\ \underline{\beta}_2 \end{bmatrix},$$

the elements of $\underline{\beta}_1^*$ are also estimable.

Proof. Let $\hat{\underline{\beta}}_1 = C\underline{Y}$ be the best linear unbiased estimator for $\underline{\beta}_1$. Then

$$\begin{aligned} E[H_1C\underline{Y}] &= H_1E[C\underline{Y}] \\ &= H_1E[\hat{\underline{\beta}}_1] \\ &= H_1\underline{\beta}_1 \\ &= \underline{\beta}_1^* . \end{aligned}$$

Since $H_1C\underline{Y}$ is unbiased for $\underline{\beta}_1^*$, the elements of $\underline{\beta}_1^*$ are estimable. The proof is complete.

Theorem 4. Let T^c be the design that results if Factor F in base plan T is collapsed to Factor F^c. If T is resolution IV with μ estimable, then the resolution of T^c is at least IV.

Proof. Theorems 1 and 2 guarantee that the conditions of Theorem 3 are satisfied. Thus if μ and the main effects of F are estimable from runs in T, μ and the main effects of F^c are estimable from the runs in T^c, the proof is complete.

The discussion of this section applies directly to the collapsing of one factor in an s^n base plan. However, repeated application of Theorems 1-4 guarantees that if T is a resolution IV s^n design with μ estimable and T^c is obtained from T by collapsing any number of factors, then T^c is also resolution IV with μ estimable.

The designs presented in the remaining sections of this paper are the smallest known resolution IV designs for the asymmetric factorial experiment. The construction technique throughout is to collapse levels of treatment combinations in the s^n foldover designs given by Anderson and Thomas (1975). In the 2^n experiment the foldover of any treatment combination is produced by a simple interchange of 0 and 1. For the 3^n experiment we extend the foldover technique by interchanging symbols in accordance with the symmetric group on three symbols $S_3 = [e, (012), (021), (01), (02), (12)]$. Thus the foldover set generated by any treatment combination $\underline{t}$ is $S_3(\underline{t}) = [\underline{t}, (012)\underline{t}, (021)\underline{t}, (01)\underline{t}, (02)\underline{t}, (12)\underline{t}]$, where, for example, $(012)\underline{t}$ is the treatment combination obtained from $\underline{t}$ by changing 0 to 1, 1 to 2, and 2 to 0. It is apparent that the foldover set of any treatment combination in $S_3(\underline{t})$ is again $S_3(\underline{t})$. The foldover set of $(0,0,\ldots,0)' = \underline{0}$ consists of three treatment combinations, $(0,0,\ldots,0)'$, $(1,1,\ldots,1)'$, and $(2,2,\ldots,2)'$, while the foldover sets of all other treatment combinations consist of six treatment combinations. Thus the 3^n treatment combinations may be partitioned into one set of size three and $(3^n - 3)/6$ sets of size six via the foldover operation.

There is a natural extension of the foldover technique to the s^n experiment employing the symmetric group on s symbols. The foldovers of any treatment combination $\underline{t}$ are obtained by making the interchanges in symbols indicated by each element of the group, and the foldover set of the element is the union of all these foldovers. The s^n treatment combinations are thus partitioned into foldover sets as in the 2^n and 3^n experiments. The foldover set of $(0,0,\ldots,0)'$ obviously contains s treatment combinations. The foldover set of any treatment combinations with only two distinct elements will have $s(s - 1)$ treatment combinations. Similarly, the foldover set of a treatment combination with k distinct elements contains $s(s - 1)(s - 2)\ldots(s - k + 1)$ treatment combinations, $k = 1, 2, \ldots, s$.

Let $\underline{\delta}_i$ denote the treatment combination with ith coordinate one and zeros elsewhere. The design T consisting of the foldover sets generated by $\underline{0}$, $\underline{\delta}_1$, $\underline{\delta}_2$, $\ldots\underline{\delta}_n$ is resolution IV in $N = s(s - 1)n + s$ runs. The design permits estimation of the mean μ, and has s degrees of freedom for estimation of error. The designs given in the remainder of the paper are obtained from this series of resolution IV designs by collapsing levels.

Since most of the existing research concerning asymmetric factorials has been directed toward the $2^m \times 3^n$ experiment, Section 3 is devoted exclusively to that case. In Section 4 the notions of Section 2 are extended to generate designs for the $s_1^m \times s_2^n$ experiment, where $s_1 < s_2$ and $s_2 > 3$. Finally, Section 5 introduces the problem of collapsing levels to produce resolution IV designs for the general asymmetric experiment.

3. The $2^m \times 3^n$ Mixed Factorial: Collapsing the 3^{m+n} Foldover Design

For the 3^{m+n} factorial experiment with $m \geq 1$, $n \geq 1$, let $T = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_N]$ represent a foldover design of the series presented in Section 2. Thus $N = 3 + 6(m + n)$. By collapsing the first m components of each $\underline{t}_i \in T$ one can obtain a design, say T^c, for a mixed factorial experiment of the type $2^m \times 3^n$, having m factors at two levels and n factors at three levels. Since T is a resolution IV design with μ estimable for the 3^{m+n} experiment, T^c will be a resolution IV design with μ estimable for the $2^m \times 3^n$ experiment.

T^c can be obtained from T according to a collapsing scheme of the type specified in (1). Suppose that the levels for each two-level factor are denoted by 0 and 1 and those for each three-level factor are denoted by 0, 1, and 2. For each $\underline{t}_i = [t_{i1}, t_{i2}, \ldots, t_{im}, t_{i,m+1}, \ldots, t_{i,m+n}]'$, $i = 1, \ldots, N$, define $\underline{t}_i^c = [t_{i1}^c, t_{i2}^c, \ldots, t_{im}^c, t_{i,m+1}, \ldots, t_{i,m+n}]'$, where for each j, $j = 1, \ldots, m$, t_{ij}^c is obtained from t_{ij} by a mapping of the form given in (6).

1. level 0 → level 0
2. level 1 → level 1 (6)
3. level 2 → level q, q = 0 or 1, selected arbitrarily.

Then $T^c = \bigcup_{i=1}^{N} \underline{t}_i^c$.

The collapsing process, of course, yields some duplication of treatment combinations for the $2^m \times 3^n$ experiment, so that N^c, the number of distinct runs in T^c, is not equal to N, the number of runs in T. Margolin's (1969a) bound for the $2^m \times 3^n$ fractional factorial experiment is $N \geq 3(m + 2n) - 3$. In order to compare the number of treatment combinations in T^c with this lower bound, it is convenient first to partition T into the three subsets T_0, T_1, and T_2, which are displayed as rows of (n + m)-component treatment combinations in

$$\begin{bmatrix} T_0 \\ \hline T_1 \\ \hline T_2 \end{bmatrix} \qquad \left[\begin{array}{c|cc|cc} \underline{0}_m & (1,0)_m & 0_{mxn} & (2,0)_m & 0_{mxn} \\ \underline{0}_n & 0_{nxm} & (1,0)_n & 0_{nxm} & (2,0)_n \\ \hline \underline{J}_m & (0,1)_m & J_{mxn} & (2,1)_m & J_{mxn} \\ \underline{J}_n & J_{nxm} & (0,1)_n & J_{nxm} & (2,1)_n \\ \hline 2\underline{J}_m & (0,2)_m & 2J_{mxn} & (1,2)_m & 2J_{mxn} \\ 2\underline{J}_n & 2J_{nxm} & (0,2)_n & 2J_{nxm} & (1,2)_n \end{array}\right] , \qquad (7)$$

where $(a,b)_k$ denotes a k x k matrix having a's on the diagonal and b's off the diagonal, $0_{k\times\ell}$ denotes a k x ℓ matrix of all zeros, $\underline{J}_k$ denotes a k x 1 vector of all ones, and $J_{k\times\ell}$ denotes a k x ℓ matrix of all ones.

Consider first the case for which $m \geq 1$ and $n > 2$.

<u>Theorem 5</u>. With $m \geq 1$ and $n > 2$, if T^c is obtained from T by a collapsing scheme of the type specified in (6), then N^c, the number of treatment combinations in T^c, equals $3(m + 2n) + 3$, which exceeds the lower bound by six.

<u>Proof</u>. Let N^c represent the number of runs in T^c and N the number of runs in T. From array (7) collapsing $T_0 \to T_0^c$, $T_1 \to T_1^c$, and $T_2 \to T_2^c$ according to a scheme of the type given in (6) produces duplication of runs in T^c as follows.

1. Since $2 \to q$ and q = 0 or 1, the collapsing $(2, j)_m \to (q, j)_m$, j = 0, 1, produces m duplicate treatment combinations in each of T_0^c and T_1^c.

2. Since $[2\underline{J}_m \mid 2\underline{J}_n]'$ in T_2 collapses to $[q\underline{J}_m \mid 2\underline{J}_n]'$ in T_2^c, where q = 0 or 1, either

$$\begin{bmatrix} (0,2)_m \\ 2J_{n\times m} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} (1,2)_m \\ 2J_{n\times m} \end{bmatrix}$$

collapses to produce m duplicates in T_2^c. Thus $N^c = N - 3m = 3 + 6(m + n) - 3m = 3(m + 2n) + 3$, which exceeds the lower bound of $3(m + 2n) - 3$ by 6. The proof is complete.

<u>Example 1</u>. In the format specified by (7) the 3^8 foldover design, which is to be collapsed to a $2^5 \times 3^3$ mixed factorial design, appears as

$$\left[\begin{array}{c|cc|cc}
\underline{0}_5 & (1,0)_5 & 0_{5x3} & (2,0)_5 & 0_{5x3} \\
\underline{0}_3 & 0_{3x5} & (1,0)_3 & 0_{3x5} & (2,0)_3 \\
\hline
\underline{J}_5 & (0,1)_5 & J_{5x3} & (2,1)_5 & J_{5x3} \\
\underline{J}_3 & J_{3x5} & (0,1)_3 & J_{3x5} & (2,1)_3 \\
\hline
2\underline{J}_5 & (0,2)_5 & 2J_{5x3} & (1,2)_5 & 2J_{5x3} \\
2\underline{J}_3 & 2J_{3x5} & (0,2)_3 & 2J_{3x5} & (1,2)_3
\end{array}\right] .$$

If q of scheme (6) is selected as 0, the collapsed design is

$$\left[\begin{array}{c|cc|cc}
\underline{0}_5 & (1,0)_5 & 0_{5x3} & (\bar{0},\bar{0})_5 & 0_{5x3} \\
\underline{0}_3 & 0_{3x5} & (1,0)_3 & 0_{3x5} & (2,0)_3 \\
\hline
\underline{J}_5 & (0,1)_5 & J_{5x3} & (\bar{0},\bar{1})_5 & J_{5x3} \\
\underline{J}_3 & J_{3x5} & (0,1)_3 & J_{3x5} & (2,1)_3 \\
\hline
\underline{0}_5 & (0,0)_5 & 0_{5x3} & (\bar{1},\bar{0})_5 & 2J_{5x3} \\
2\underline{J}_3 & 2J_{3x5} & (0,2)_3 & 2J_{3x5} & (1,2)_3
\end{array}\right] .$$

with duplicate treatment combinations indicated by dotted line enclosures. Thus the design for the $2^5 \times 3^3$ experiment consists of $N^c = 36$ treatment combinations.

In the special cases for which $n = 1$ or $n = 2$ collapsing scheme (6) produces some duplication in T^c in addition to that provided by Theorem 5. The results for these cases are summarized in Theorem 6.

<u>Theorem 6</u>. Suppose T is a 3^{m+n} foldover design and T^c is a $2^m \times 3^n$ design obtained from T by a collapsing scheme of type (6). Let N^c denote the number of treatment combinations in T^c.

1. If $n = 1$, $N^c = 3(m + 2)$, which exceeds the lower bound for $2^m \times 3$ factorial designs by 3.
2. If $n = 2$, $N^c = 3m + 14$, which exceeds the lower bound for $2^m \times 3^2$ factorial designs by 4.

<u>Proof</u>. 1: If $n = 1$, array (7) becomes

$$\left[\begin{array}{c} T_0 \\ \hline T_1 \\ \hline T_2 \end{array}\right] = \left[\begin{array}{c|cc|cc} \underline{0}_m & (1,0)_m & \underline{0}_m & (2,0)_m & \underline{0}_m \\ 0 & 0 \dots 0 & 1 & 0 \dots 0 & 2 \\ \hline \underline{J}_m & (0,1)_m & \underline{J}_m & (2,1)_m & \underline{J}_m \\ 1 & 1 \dots 1 & 0 & 1 \dots 1 & 2 \\ \hline 2\underline{J}_m & (0,2)_m & 2\underline{J}_m & (1,2)_m & 2\underline{J}_m \\ 2 & 2 \dots 2 & 0 & 2 \dots 2 & 1 \end{array}\right]$$

which collapses to

$$\left[\begin{array}{c} T_0^c \\ \hline T_1^c \\ \hline T_2^c \end{array}\right] = \left[\begin{array}{c|cc|cc} \underline{0}_m & (1,0)_m & \underline{0}_m & (q,0)_m & \underline{0}_m \\ 0 & 0 \dots 0 & 1 & 0 \dots 0 & 2 \\ \hline \underline{J}_m & (0,1)_m & \underline{J}_m & (q,1)_m & \underline{J}_m \\ 1 & 1 \dots 1 & 0 & 1 \dots 1 & 2 \\ \hline q\underline{J}_m & (0,q)_m & q\underline{J}_m & (1,q)_m & q\underline{J}_m \\ 2 & 2 \dots 2 & 0 & 2 \dots 2 & 2 \end{array}\right] .$$

As in Theorem 5, each of T_0^c and T_1^c includes exactly m duplicate treatment combinations. However, since n = 1 *and every treatment combination* of the form $[q\underline{J}_m \mid r]'$, where r = 0 or 2, appears in T_0^c or T_1^c, all but m of the treatment combinations in T_2^c are duplicates. Thus, if n = 1, $N^c = 3(m + 2)$, which exceeds the theoretical lower bound by only three runs.

2: A similar argument holds when n = 2 in which case

$$\left[\begin{array}{c} T_0 \\ \hline T_1 \\ \hline T_2 \end{array}\right] = \left[\begin{array}{c|ccc|ccc} \underline{0}_m & (1,0)_m & \underline{0}_m & \underline{0}_m & (2,0)_m & \underline{0}_m & \underline{0}_m \\ 0 & 0 \dots 0 & 1 & 0 & 0 \dots 0 & 2 & 0 \\ 0 & 0 \dots 0 & 0 & 1 & 0 \dots 0 & 0 & 2 \\ \hline \underline{J}_m & (0,1)_m & \underline{J}_m & \underline{J}_m & (2,1)_m & \underline{J}_m & \underline{J}_m \\ 1 & 1 \dots 1 & 0 & 1 & 1 \dots 1 & 2 & 1 \\ 1 & 1 \dots 1 & 1 & 0 & 1 \dots 1 & 1 & 2 \\ \hline 2\underline{J}_m & (0,2)_m & 2\underline{J}_m & 2\underline{J}_m & (1,2)_m & 2\underline{J}_m & 2\underline{J}_m \\ 2 & 2 \dots 2 & 0 & 2 & 2 \dots 2 & 1 & 2 \\ 2 & 2 \dots 2 & 2 & 0 & 2 \dots 2 & 2 & 1 \end{array}\right]$$

collapses to

$$\begin{bmatrix} T_0^c \\ \hline T_1^c \\ \hline T_2^c \end{bmatrix} = \left[\begin{array}{c|cccc|ccc} \underline{0}_m & (1,0)_m & \underline{0}_m & \underline{0}_m & (q,0)_m & \underline{0}_m & \underline{0}_m \\ 0 & 0 \ldots 0 & 1 & 0 & 0 \ldots 0 & 2 & 0 \\ 0 & 0 \ldots 0 & 0 & 1 & 0 \ldots 0 & 0 & 2 \\ \hline \underline{J}_m & (0,1)_m & \underline{J}_m & \underline{J}_m & (q,1)_m & \underline{J}_m & \underline{J}_m \\ 1 & 1 \ldots 1 & 0 & 1 & 1 \ldots 1 & 2 & 1 \\ 1 & 1 \ldots 1 & 1 & 0 & 1 \ldots 1 & 1 & 2 \\ \hline q\underline{J}_m & (0,q)_m & q\underline{J}_m & q\underline{J}_m & (1,q)_m & q\underline{J}_m & q\underline{J}_m \\ 2 & 2 \ldots 2 & 0 & 2 & 2 \ldots 2 & 1 & 2 \\ 2 & 2 \ldots 2 & 2 & 0 & 2 \ldots 2 & 2 & 1 \end{array}\right]$$

thus producing

$$\begin{bmatrix} q\underline{J}_m & q\underline{J}_m \\ 0 & 2 \\ 2 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} q\underline{J}_m & q\underline{J}_m \\ 1 & 2 \\ 2 & 1 \end{bmatrix}$$

as duplicate observations in addition to those guaranteed by Theorem 5. The proof is complete.

Example 2. If the 3^8 foldover design of Example 1 is collapsed by scheme (6) with q chosen as 0 to a $2^7 \times 3$ mixed factorial design, the result is

$$\left[\begin{array}{cccc|c} \underline{0}_7 & (1,0)_7 & \underline{0}_7 & (0,0)_7 & \underline{0}_7 \\ 0 & 0 \ldots 0 & 1 & 0 \ldots 0 & 2 \\ \hline \underline{J}_7 & (0,1)_7 & \underline{J}_7 & (0,1)_7 & \underline{J}_7 \\ 1 & 1 \ldots 1 & 0 & 1 \ldots 1 & 2 \\ \hline \underline{0}_7 & (0,0)_7 & \underline{0}_7 & (1,0)_7 & \underline{0}_7 \\ 2 & 2 \ldots 2 & 0 & 2 \ldots 2 & 2 \end{array}\right]$$

with duplicate treatment combinations indicated by dotted line enclosures. Thus $N^c = 27$.

4. The $s_1^m \times s_2^n$ Mixed Factorial: Collapsing the s_2^{m+n} Foldover Design

The results of Section 2 extend readily to the problem of collapsing levels of the treatment combinations in an s_2^{m+n} foldover design to produce an $s_1^m \times s_2^n$ mixed factorial design. Suppose specifically $s_2 > 3$, $s_1 < s_2$ and that the levels for each of the first m factors are labelled 0, 1, ..., $s_1 - 1$ while those for each of the remaining factors are labelled 0, 1, ..., $s_2 - 1$. As in Section 2, $T = [\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_N]$ represents the foldover design for the s_2^{m+n} experiment, and $T^c = [\underline{t}_1^c, \underline{t}_2^c, \ldots, \underline{t}_{N^c}^c]$ represents the design for the $s_1^m \times s_2^n$ experiment which results from the collapsing process. For each $\underline{t}_i = [t_{i1}, \ldots, t_{im}, t_{i,m+1}, \ldots, t_{i,m+n}]'$, i = 1, ..., N, *define* $\underline{t}_i^c = [t_{i1}^c, t_{i2}^c, \ldots, t_{im}^c, t_{i,m+1}, \ldots, t_{i,m+n}]'$ where for each j, j = 1, ..., m, t_{ij}^c is obtained from t_{ij} by a mapping of the type specified in (8).

For levels 0, ..., $s_1 - 1$: level $\ell \to$ level ℓ (8)

For levels $\ell_1 = s_1, \ldots, \ell_{s_2-s_1} = s_2 - 1$: level $\ell_j \to$ level q_j, *where* q_j *is an arbitrarily selected member of* $\{0,1,\ldots,s_1-1\}$.

Then $T^c = \bigcup_{i=1}^{N} \underline{t}_i^c$.

In order to investigate the number of runs resulting from application of collapsing scheme (8), it is convenient to arrange the treatment combinations of the s_2^{m+n} foldover design in s_2 rows as shown in Table 1.

$$
\begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_{s_1-1} \\ T_{s_1} \\ T_{s_1+1} \\ \vdots \\ T_{s_2-1} \end{bmatrix}
=
\left[\begin{array}{l|lll|llll}
\underline{0}_{(m+n)} & (1,0) & \cdots & (s_1-1,0) & (s_1,0) & & (s_1+1,0) & \ldots(s_2-1,0) \\
\underline{J}_{(m+n)} & (0,1) & \cdots & (s_1-1,1) & (s_1,1) & & (s_1+1,1) & \ldots(s_2-1,1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
(s_1-1)\underline{J}_{(m+n)} & (0,s_1-1) & \cdots & (s_1-2,s_1-1) & & & & \\
\hline
s_1\underline{J}_{(m+n)} & (0,2_1) & \cdots & (s_1-2,s_1) & (s_1-1,s_1) & & (s_1+1,s_1) & \ldots(s_2-1,s_1) \\
(s_1+1)\underline{J}_{(m+n)} & (0,s_1+1) & \cdots & (s_1-2,s_1+1) & (s_1-1,s_1+1) & & (s_1,s_1+1) & \ldots(s_2-1,s_1+1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\
(s_2-1)\underline{J}_{(m+n)} & (0,s_2-1) & \cdots & (s_1-2,s_2-1) & (s_1-1,s_2-1) & & (s_1,s_2-1) & \ldots(s_2-2,s_2-1)
\end{array}\right]
$$

Table 1. Treatment Combinations in the s_2^{m+n} Foldover Design

In Table 1 each (a,b) denotes an $(m + n) \times (m + n)$ matrix with a's on the diagonal and b's off the diagonal, $\underset{\sim}{0}_{m+n}$ denotes an $(m + n)$-component vector of all zeros, and $\underset{\sim}{J}_{m+n}$ denotes an $(m + n)$-component vector of all ones.

The s_2^{m+n} foldover design displayed in Table 1 consists of $N = s_2 + s_2(s_2 - 1)(m + n)$ treatment cominations, thus exceeding the theoretical lower bound by $s_2(s_2 - 1)$ runs. For the $s_1^m \times s_2^n$ factorial experiment, $s_1 < s_2$, the number of treatment combinations in a resolution IV design must satisfy

$$N \geq s_2[(s_1 - 1)m + (s_2 - 1)n - (s_2 - 2)]. \tag{9}$$

The following theorems provide a comparison of N^c, the number of runs in T^c, with the lower bound (9).

Consider first the case in which $m \geq 1$ and $n > 2$.

<u>Theorem 7.</u> If T is a foldover design for the s_2^{m+n} experiment, $m \geq 1$, $n > 2$, and T^c is obtained from T by any collapsing scheme of type (8), then N^c exceeds the lower bound (9) by $s_2(s_2 - 1)$ runs.

<u>Proof</u>. Starting with T as displayed in Table 1, collapse each $T_j \rightarrow T_j^c$, $j = 0, \ldots, s_2 - 1$, according to any scheme of the type given in (8). The argument in the proof of Theorem 5 extends easily to guarantee that $(s_2 - s_1)a$ duplicate treatment combinations occur in each T_j^c, $j = 1, \ldots, m$. Specifically, (10) below shows the number of treatment combinations that can be eliminated from various subsets of T^c, partitioned in accordance with the heavy lines of Table 1.

$$\left[\begin{array}{c|c} 0 & s_1(s_2-s_1)m \\ \hline (s_2-s_1)m & (s_2-s_1)(s_2-s_1-1)m \end{array}\right] \tag{10}$$

Since the number of runs in the s_2^{m+n} foldover design is

$$\begin{aligned} N &= s_2 + s_2(s_2 - 1)(m + n) \\ &= s_2 + s_2(s_2 - 1)m + s_2(s_2 - 1)n \\ &= s_2 + s_2[(s_2 - s_1) + (s_1 - 1)]m + s_2(s_2 - 1)n, \end{aligned}$$

it follows that

$$\begin{aligned} N^c &= s_2 + s_2(s_2 - s_1)m + s_2(s_1 - 1)m + s_2(s_2 - 1)n - s_2(s_2 - s_1)m \\ &= s_2[1 + (s_1 - 1)m + (s_2 - 1)n]. \end{aligned}$$

Thus N^c exceeds the theoretical lower bound given in (9) by $s_2(s_2 - 1)$ runs. The proof is complete.

<u>Example 3.</u> Let T be a 5^{m+n}, $n > 2$, foldover design which is to be collapsed to a $2^m \times 5^n$ mixed factorial design, denoted by T^c. Under the collapsing scheme of type (8) that maps level 2 → level 0, level 3 → level 1, and level 4 → level 1, T^c is as shown in (11) with duplicate treatment combinations indicated by dotted line enclosures.

$$\left[\begin{array}{ccccccccc}
\underline{0}_m & (1,0)_m & 0_{m\times n} & (0,0)_m & 0_{m\times n} & (1,0)_m & 0_{m\times n} & (1,0)_m & 0_{m\times n} \\
\underline{0}_n & 0_{n\times m} & (1,0)_n & 0_{n\times m} & (2,0)_n & 0_{n\times m} & (3,0)_n & 0_{n\times m} & (4,0)_n \\
\hline
\underline{J}_m & (0,1)_m & J_{m\times n} & (0,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} \\
\underline{J}_n & J_{n\times m} & (0,1)_n & J_{n\times m} & (2,1)_n & J_{n\times m} & (3,1)_n & J_{n\times m} & (4,1)_n \\
\hline
\underline{0}_m & (0,0)_m & 0_{m\times n} & (1,0)_m & 0_{m\times n} & (1,0)_m & 0_{m\times n} & (1,0)_m & 0_{m\times n} \\
2\underline{J}_n & 2J_{n\times m} & (0,2)_n & 2J_{n\times m} & (1,2)_n & 2J_{n\times m} & (3,2)_n & 2J_{n\times m} & (4,2)_n \\
\hline
\underline{J}_m & (0,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} & (0,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} \\
3\underline{J}_n & 3J_{n\times m} & (0,3)_n & 3J_{n\times m} & (1,3)_n & 3J_{n\times m} & (2,3)_n & 3J_{n\times m} & (4,3)_n \\
\hline
\underline{J}_m & (0,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} & (0,1)_m & J_{m\times n} & (1,1)_m & J_{m\times n} \\
4\underline{J}_n & 4J_{n\times m} & (0,4)_n & 4J_{n\times m} & (1,4)_n & 4J_{n\times m} & (2,4)_n & 4J_{n\times m} & (3,4)_n
\end{array}\right] \tag{4.11}$$

In this case $N^c = 5(1 + m + 4n)$, which exceeds the lower bound $5(-3 + m + 4n)$ by exactly 20 runs.

The cases for which n = 1 and n = 2 are considered in Theorem 8.

Theorem 8. Suppose T is an s_2^{m+n} foldover design and T^c is an $s_1^m \times s_2^n$ design obtained from T by a collapsing scheme of type (8).

1. If n = 1, $N^c = s_2[(s_1 - 1)m + s_1]$, which exceeds the lower bound by $s_2(s_1 - 1)$ runs.

2. If n = 2, $N^c = s_2[-3 + (s_1 - 1)m + 2s_2] + 2s_1$, which exceeds the lower bound by $s_2^2 - 3s_2 + 2s_1$ runs.

Proof. If n = 1 and T is collapsed according to a scheme of type (8), each T_j^c, $j = 0, \ldots, s_1 - 1$, includes exactly $(s_2 - s_1)m$ duplicate treatment combinations. However, every treatment combination of the form $[q\underline{J}_m \mid r]'$, $q = s_1, \ldots, s_2 - 1$, necessarily collapses to a treatment combination that appears in T_j^c for some j, $j = 0,1,\ldots,s_1-1$.

Thus for $j = s_1, \ldots, s_2 - 1$, each T_j^c includes a total of $(s_2 - s_1)m + s_2$ duplicate treatment combinations. Therefore, if $n = 1$,

$$N^c = s_2[1 + (s_1 - 1)m + (s_2 - 1)] - s_2(s_2 - s_1)$$
$$= s_2[(s_1 - 1)m + (s_2 - 1) + (1 - s_2 + s_1)],$$

which exceeds the lower bound (9) by $s_2(s_1 - 1)$.

2: A similar argument holds when $n = 2$ in which case each T_j^c, $j = s_1, \ldots, s_2 - 1$, includes a total of $(s_2 - s_1)m + 2$ duplicate treatment combinations. Then $N^c = s_2[1 + (s_1 - 1)m + 2(s_2 - 1]$ $- 2(s_2 - s_1)$, which exceeds the lower bound by $s_2^2 - 3s_2 + 2s_1$.

4. The $s_1^{n_1} \times s_2^{m_2} \times \ldots \times s_k^{n_k}$ Mixed Factorial Experiment

If a foldover design, T, for the $s_k^{n_1 + \ldots + n_k}$ experiment is to be collapsed to produce a design, T^c, for the $s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k}$ mixed factorial experiment, the problem of determining N^c becomes more complex. It is therefore instructive to consider first an $s_3^{n_1+n_2+n_3}$ foldover design, which is to be collapsed to an $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ mixed factorial design in which $s_1 < s_2 < s_3$. If $T = [\underline{t}_1, \ldots, \underline{t}_N]$ represents the s_3^n foldover design, then $N = s_3 + s_3(s_3 - 1)(n_1+n_2+n_3)$. For each $\underline{t}_i = [t_{i1}, \ldots, t_{in_1}, \ldots, t_{i,n_1+n_2}, \ldots, t_{i,n_1+n_2+n_3}]'$, $i = 1, \ldots, N$, define $\underline{t}_i^c = [t_{i1}^c, \ldots, t_{i,n_1+n_2}^c, t_{i,n_1+n_2+1}, \ldots, t_{i,n_1+n_2+n_3}]$, where t_{ij}^c is determined from t_{ij} according to a collapsing scheme as specified in (4.12).

Case 1: $j = 1, \ldots, n_1$

For levels $0, \ldots, s_1 - 1$: level $\ell \rightarrow$ level ℓ

For levels $\ell_1 = s_1, \ldots, \ell_{s_2-s_1} = s_2 - 1, \ldots, \ell_{s_3-s_1}$ $= s_3 - 1$: level $\ell_j \to$ level q_j, where q_j is an arbitrarily selected member of $\{0, 1, \ldots, s_1 - 1\}$.

Case 2: $j = n_1 + 1, \ldots, n_1 + n_2$

For levels $0, \ldots, s_2 - 1$: level $\ell \to$ level ℓ

For levels $\ell_{s_2-s_1+1} = s_2, \ldots, \ell_{s_3-s_1} = s_3 - 1$: level $\ell_j \to$ level q_j^*, where q_j^* is an arbitrarily selected member of $\{0, 1, \ldots, s_2 - 1\}$.

Then $T^c = \bigcup_{i=1}^{N} \underline{t}_i^c$.

As in preceding sections, it is convenient to arrange the treatment combinations of the s_3^n foldover design as shown in Table 2. Each (a, b) of Table 2 represents an $(n_1 + n_2 + n_3) \times (n_1 + n_2 + n_3)$ matrix with a's on the diagonal and b's off the diagonal, which can be partitioned as shown in (13).

$$(a,b) = \begin{bmatrix} (a,b)_{n_1} & bJ_{n_1 \times n_2} & bJ_{n_1 \times n_3} \\ bJ_{n_2 \times n_1} & (a,b)_{n_2} & bJ_{n_2 \times n_3} \\ bJ_{n_3 \times n_1} & bJ_{n_3 \times n_2} & (a,b)_{n_3} \end{bmatrix} \tag{13}$$

Theorem 9, which follows, provides a basis for comparison of N^c, the number of runs in T^c, with the lower bound for asymmetric $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ designs, which is given by

$$N \geq s_3[(s_1 - 1)n_1 + (s_2 - 1)n_2 + (s_3 - 1)n_3 - (s_3 - 2)]. \tag{14}$$

$$
\begin{bmatrix} T_0 \\ T_1 \\ \vdots \\ T_{s_1-1} \\ T_{s_1} \\ T_{s_1+1} \\ \vdots \\ T_{s_2-1} \\ T_{s_2} \\ T_{s_2+1} \\ \vdots \\ T_{s_3-1} \end{bmatrix}
\begin{bmatrix}
\underline{0}_n & (1,0) & \cdots & (s_1-1,0) & (s_1,0) & \cdots & (s_2-1,0) & (s_2,0) & \cdots & (s_3-1,0) \\
\underline{J}_n & (0,1) & \cdots & (s_1-1,1) & (s_1,1) & \cdots & (s_2-1,1) & (s_2,1) & \cdots & (s_3-1,1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
(s_1-1)\underline{J}_n & (0,s_1-1) & \cdots & (s_1-2,s_1-1) & (s_1,s_1-1) & \cdots & (s_2-1,s_1-1) & (s_2,s_2-1) & \cdots & (s_3-1,s_1-1) \\
\hline
s_1\underline{J}_n & (0,s_1) & \cdots & (s_1-2,s_1) & (s_1-1,s_1) & \cdots & (s_2-1,s_1) & (s_2,s_1) & \cdots & (s_3-1,s_1) \\
(s_1+1)\underline{J}_n & (0,s_1+1) & \cdots & (s_1-2,s_1+1) & (s_1-1,s_1+1) & \cdots & (s_2-1,s_1+1) & (s_2,s_1+1) & \cdots & (s_3-1,s_1+1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
(s_2-1)\underline{J}_n & (0,s_2-1) & \cdots & (s_1-2,s_2-1) & (s_1-1,s_2-1) & \cdots & (s_2-2,s_2-1) & (s_2,s_2-1) & \cdots & (s_3-1,s_2-1) \\
\hline
s_2\underline{J}_n & (0,s_2) & \cdots & (s_1-2,s_2) & (s_1-1,s_2) & \cdots & (s_2-2,s_2) & (s_2-1,s_2) & \cdots & (s_3-1,s_2) \\
(s_2+1)\underline{J}_n & (0,s_2+1) & \cdots & (s_1-2,s_2+1) & (s_1-1,s_2+1) & \cdots & (s_2-2,s_2+1) & (s_2-1,s_2+1) & \cdots & (s_3-1,s_2+1) \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
(s_3-1)\underline{J}_n & (0,s_3-1) & \cdots & (s_1-2,s_3-1) & (\,_1-1,s_3-1) & \cdots & (s_2-2,s_3-1) & (s_2-1,s_3-1) & \cdots & (s_3-1,s_3-1)
\end{bmatrix}
$$

Table 2. Treatment Combinations in the s_3^n Foldover Design

Consider first the case in which $n_1 \geq 1$, $n_2 \geq 1$, and $n_3 > 2$.

Theorem 9. If T is a foldover design for the $s_3^{n_1+n_2+n_3}$ experiment, $n_1 \geq 1$, $n_2 \geq 1$, and $n_3 > 2$, and T^c is an $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ mixed factorial design obtained from T by any collapsing scheme of type (12). then N^c exceeds the lower bound (14) by $s_3(s_3 - 1)$ runs.

Proof. Starting with T as shown in Table 2, collapse each $T_j \to T_j^c$, $j = 0, \ldots, s_3 - 1$, according to any scheme of the type given in (12). Theorem 7 of Section 3, applied twice, guarantees that duplicate treatment combinations can be eliminated from T^c in accordance with (15), which is partitioned in the same way as Table

$$\left[\begin{array}{c|c|c} 0 & s_1(s_2-s_1)n_1 & s_1(s_3-s_2)(n_1+n_2) \\ \hline (s_2-s_1)n_1 & (s_2-s_1)(s_2-s_1-1)n_1 & (s_2-s_1)(s_3-s_2)(n_1+n_2) \\ \hline \multicolumn{2}{c|}{(s_3-s_2)[(s_2-s_1+1)n_1 + n_2]} & (s_3-s_2)(s_3-s_2-1)(n_1+n_2) \end{array}\right] \tag{15}$$

Thus each T_j^c, $j = 0, \ldots, s_3 - 1$, includes $\sum_{i=1}^{2} (s_3 - s_i)n_i$ duplicates, so a total of $s_3[(s_3 - s_1)n_1 + (s_3 - s_2)n_2]$ runs can be eliminated from T^c. Since the number of runs in the $s_3^{n_1+n_2+n_3}$ foldover design is

$$\begin{aligned} N &= s_3 + s_3(s_3 - 1)(n_1 + n_2 + n_3) \\ &= s_3 + s_3[(s_3 - s_1) + (s_1 - 1)]n_1 + s_3[(s_3 - s_2) + (s_2 - 1)]n_2 \\ &\quad + s_3(s_3 - 1)n_3, \end{aligned}$$

it follows by subtraction that

$$\begin{aligned} N^c &= s_3 + s_3(s_1 - 1)n_1 + s_3(s_2 - 1)n_2 + s_3(s_3 - 1)n_3 \\ &= s_3[1 + (s_1 - 1)n_1 + (s_2 - 1)n_2 + (s_3 - 1)n_3], \end{aligned}$$

which exceeds lower bound (14) by $s_3(s_3-1)$ runs. The proof is complete.

<u>Example 4.</u> Suppose a $5^{n_1+n_2+n_3}$, $n_3 \geq 2$, foldover design is collapsed to a $2^{n_1} \times 3^{n_2} \times 5^{n_3}$ asymmetric factorial design under the collapsing scheme (16).

$$\begin{array}{ll} \text{For the } n_1 \text{ factors at two levels:} & \text{level } 2 \rightarrow \text{level } 0 \\ & \text{level } 3 \rightarrow \text{level } 1 \\ & \text{level } 4 \rightarrow \text{level } 0 \\ \text{For the } n_2 \text{ factors at three levels:} & \text{level } 3 \rightarrow \text{level } 0 \\ & \text{level } 4 \rightarrow \text{level } 2 \end{array} \tag{16}$$

T^c is shown in (17) with duplicate treatment combinations indicated by dotted line enclosures. Appropriate dimensions are indicated by subscripts in the first column.

$$\left[\begin{array}{c|ccc|ccc|ccc|ccc}
0 & (1,0) & 0 & 0 & (0,0) & 0 & 0 & (1,0) & 0 & 0 & (0,0) & 0 & 0 \\
\overline{0}^{n_1} & 0 & (1,0) & 0 & 0 & (2,0) & 0 & 0 & (0,0) & 0 & 0 & (2,0) & 0 \\
\overline{0}^{n_2} & 0 & 0 & (1,0) & 0 & 0 & (2,0) & 0 & 0 & (3,0) & 0 & 0 & (4,0) \\
\overline{}^{n_3} & & & & & & & & & & & & \\
J & (0,1) & J & J & (0,1) & J & J & (1,1) & J & J & (0,1) & J & J \\
\overline{J}^{n_1} & J & (0,1) & J & J & (2,1) & J & J & (0,1) & J & J & (2,1) & J \\
\overline{J}^{n_2} & J & J & (0,1) & J & J & (2,1) & J & J & (3,1) & J & J & (4,1) \\
\overline{}^{n_3} & & & & & & & & & & & & \\
0 & (0,0) & 0 & 0 & (1,0) & 0 & 0 & (1,0) & 0 & 0 & (0,0) & 0 & 0 \\
2\overline{J}^{n_1} & 2J & (0,2) & 2J & 2J & (1,2) & 2J & 2J & (0,2) & 2J & 2J & (2,2) & 2J \\
2\overline{J}^{n_2} & 2J & 2J & (0,2) & 2J & 2J & (1,2) & 2J & 2J & (3,2) & 2J & 2J & (4,2) \\
\overline{}^{n_3} & & & & & & & & & & & & \\
J & (0,1) & J & J & (1,1) & J & J & (0,1) & J & J & (0,1) & J & J \\
\overline{0}^{n_1} & 0 & (0,0) & 0 & 0 & (1,0) & 0 & 0 & (2,0) & 0 & 0 & (2,0) & 0 \\
3\overline{J}^{n_2} & 3J & 3J & (0,3) & 3J & 3J & (1,3) & 3J & 3J & (2,3) & 3J & 3J & (4,3) \\
\overline{}^{n_3} & & & & & & & & & & & & \\
0 & (0,0) & 0 & 0 & (1,0) & 0 & 0 & (0,0) & 0 & 0 & (1,0) & 0 & 0 \\
2\overline{J}^{n_1} & 2J & (0,2) & 2J & 2J & (1,2) & 2J & 2J & (2,2) & 2J & 2J & (0,2) & 2J \\
4\overline{J}^{n_2} & 4J & 4J & (0,4) & 4J & 4J & (1,4) & 4J & 4J & (2,4) & 4J & 4J & (3,4) \\
\overline{}^{n_3} & & & & & & & & & & & &
\end{array}\right] \quad (17)$$

Thus $N^c = 5(1 + n_1 + 2n_2 + 4n_3)$.

If $n_3 = 1$ or $n_3 = 2$, duplicate treatment combinations in addition to those guaranteed by Theorem 9 may arise. However, the exact number of additional duplicates depends on the particular choice of collapsing scheme (12). Specifically, the amount of duplication is affected in a rather complex way by how often $q_j = q_j^*$, $j = s_2-s_1+1, \ldots s_3-1$, by whether each q_j^* is selected from $\{0, \ldots, s_1 - 1\}$ or from $\{s_1, \ldots, s_2 - 1\}$, and by how often $q_j = q_{j'}$ and $q_j^* = q_{j'}^*$ for any pair of levels ℓ_j and $\ell_{j'}$ with $j \neq j'$. However, one simple result deserves mention. If $n_3 = 1$, and a collapsing scheme is selected so that for $j = s_2 - s_1 + 1, \ldots, s_3 - 1$, $\ell_j \rightarrow q_j = q_j^*$, where $q_j = q_j^*$ is an arbitrarily selected member of $\{0, 1, \ldots, s_1 - 1\}$, then T^c includes an additional $(s_3 - s_2)(s_3)$ duplicate treatment combinations. This result is illustrated in Example 5.

Example 4.5. Suppose the $5^{n_1+n_2+1}$ foldover design is collapsed to a $2^{n_1} \times 3^{n_2} \times 5$ asymmetric factorial design by collapsing scheme (18).

$$\begin{array}{lll}
\text{For the } n_1 \text{ factors at two levels:} & \text{level } 2 \rightarrow \text{level } 0 & \\
 & \text{level } 3 \rightarrow \text{level } 1 & (18) \\
 & \text{level } 4 \rightarrow \text{level } 0 & \\
\text{For the } n_2 \text{ factors at three levels:} & \text{level } 3 \rightarrow \text{level } 1 & \\
 & \text{level } 4 \rightarrow \text{level } 0 &
\end{array}$$

The resulting design is shown in (19) with duplicate treatment combinations indicated by dotted line enclosures. Appropriate dimensions are indicated by subscripts in the first column.

0	(1,0)	0	0	(0,0)	0	0	(1,0)	0	0	(0,0)	0	0			
$\overline{0}^{n_1}$	0	(1,0)	$\overline{0}$	0	(2,0)	$\overline{0}$	0	(1,0)	$\overline{0}$	0	(0,0)	$\overline{0}$			
$\overline{0}^{n_2}$	0	0	$\overline{1}$	0	0	$\overline{2}$	0	0	$\overline{3}$	0	0	$\overline{4}$			
J	(0,1)	J	J	(0,1)	J	J	(1,1)	J	J	(0,1)	J	J			
$\overline{J}^{n_1}$	J	(0,1)	$\overline{J}$	J	(2,1)	$\overline{J}$	J	(1,1)	$\overline{J}$	J	(0,1)	$\overline{J}$			
$\overline{1}^{n_2}$	J	J	$\overline{0}$	J	J	$\overline{2}$	J	J	$\overline{3}$	J	J	$\overline{4}$			
0	(0,0)	0	0	(1,0)	0	0	(1,0)	0	0	(0,0)	0	0			
$\overline{2J}^{n_1}$	2J	(0,2)	$\overline{2J}$	2J	(1,2)	$\overline{2J}$	2J	(1,2)	$\overline{2J}$	2J	(0,2)	$\overline{2J}$			(19)
$\overline{2}^{n_2}$	2J	2J	$\overline{0}$	2J	2J	$\overline{1}$	2J	2J	$\overline{3}$	2J	2J	$\overline{4}$			
J	(0,1)	J	J	(1,1)	J	J	(0,1)	J	J	(0,1)	J	J			
$\overline{J}^{n_1}$	J	(0,1)	$\overline{J}$	J	(1,1)	$\overline{J}$	J	(2,1)	$\overline{J}$	J	(0,1)	$\overline{J}$			
$\overline{3}^{n_2}$	3J	3J	$\overline{0}$	3J	3J	$\overline{1}$	3J	3J	$\overline{2}$	3J	3J	$\overline{4}$			
0	(0,0)	0	0	(1,0)	0	0	(0,0)	0	0	(1,0)	0	0			
$\overline{0}^{n_1}$	0	(0,0)	$\overline{0}$	0	(1,0)	$\overline{0}$	0	(2,0)	$\overline{0}$	0	(1,0)	$\overline{0}$			
$\overline{4}^{n_2}$	4J	4J	$\overline{0}$	4J	4J	$\overline{1}$	4J	4J	$\overline{2}$	4J	4J	$\overline{3}$			

Thus $N^c = 5n_1 + 10n_2 + 15$, which exceeds the lower bound by 10 runs.

For $k > 3$ the problem of collapsing a foldover design for the $s_k^{n_1+\ldots+n_k}$ experiment to create a design for the $s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k}$ asymmetric experiment, $s_1 < s_2 < \ldots < s_k$, becomes increasingly complicated. For each $\underline{t}_i = [t_{i1}, \ldots, t_{i,n_1+\ldots n_k}]'$, $i = 1, \ldots, N$, define $\underline{t}_i^c = [t_{i1}^c, \ldots, t_{i,n_1+\ldots+n_{k-1}}^c \ldots t_{i,n_1+\ldots+n_k}]'$, where t_{ij}^c is determined from t_{ij} according to a collapsing scheme as specified in (20).

Case 1: $j = 1, \ldots, n_1$ (20)

For levels $0, \ldots, s_1 - 1$: level $\ell \rightarrow$ level ℓ

For levels $\ell_1 = s_1, \ldots, \ell_{s_k - s_1} = s_k - 1$: level $\ell_j \rightarrow q_{j_1}$, where q_{j_1} is an arbitrarily selected member of $\{0, 1, \ldots, s_1 - 1\}$

Case m (for $m = 2, \ldots, k - 1$): $j = n_{m-1} + 1, \ldots, n_{m-1} + n_m$

For levels $0, \ldots, s_m - 1$: level $\ell \rightarrow$ level ℓ

For levels $\ell_{s_m - s_1 + 1} = s_m, \ldots, \ell_{s_k - s_1} = s_k - 1$: level $\ell_j \rightarrow$ level q_{jm}, where q_{jm} is an arbitrarily selected member of $\{0, 1, \ldots, s_m - 1\}$.

If T^c is obtained from T by a collapsing scheme of type (20), the exact number of duplicate treatment combinations in T^c depends on n_k and on the particular choice of collapsing pattern. However, Theorem 10 provides a lower bound for the number of duplicate runs, thus providing a basis for comparison of N^c and the theoretical lower bound

$$N \geq s_k[\sum_1^k (s_i - 1)n_i - (s_k - 2)]. \quad (21)$$

Theorem 10. If T is a foldover design for the $s_k^{n_1+\ldots+n_k}$ experiment and T^c is an $s_1^{n_1} \times s_2^{n_2} \times \ldots \times s_k^{n_k}$ mixed factorial design obtained from T by any collapsing scheme of type (20), then N^c exceeds the lower bound (21) by no more than $s_k(s_k - 1)$ runs.

<u>Proof</u>. Let T_j^c, $j = 0, \ldots, s_3 - 1$, be defined as in Section 3. Then repeated application of Theorem 7 guarantees that each T_j^c includes at least $\sum_{i=1}^{k-1} (s_3 - s_i)n_i$ duplicates, so a total of $s_k \left[\sum_{i=1}^{k-1} (s_k - s_i)n_i\right]$ treatment combinations can be eliminated from T^c. Since the number of runs in the $s_k^{n_1+\cdots+n_k}$ foldover design is

$$N = s_k + s_k(s_k - 1)\left(\sum_1^k n_i\right)$$

$$= s_k + \sum_1^{k-1} s_k[(s_k - s_i) + (s_i - 1)]n_i + s_k(s_k - 1)n_k,$$

it follows by subtraction that

$$N^c \leq s_k\left[1 + \sum_1^k (s_i - 1)n_i\right],$$

which exceeds lower bound (4.21) by $s_k(s_k - 1)$ runs. The proof is complete.

<u>Example 6</u>. Suppose a $5^{n_1+n_2+n_3+2}$ foldover design is collapsed to a $2^{n_1} \times 3^{n_2} \times 4^{n_3} \times 5^2$ asymmetric factorial design by means of mapping (22).

$$\begin{array}{l} \text{level } 2 \rightarrow \text{level } 0 \\ \text{level } 3 \rightarrow \text{level } 1 \\ \text{level } 4 \rightarrow \text{level } 1 \end{array} \tag{22}$$

T^c is shown in (23) with duplicate treatment combinations indicated by dotted line enclosures. Appropriate dimensions are indicated by subscripts in the first column.

0	(1,0)	0	0	0	(0,0)	0	0	0	(1,0)	0	0	0	(1,0)	0	0	0
0 n_1	0	(1,0)	0	0	0	(2,0)	0	0	0	(1,0)	0	0	0	(1,0)	0	0
0 n_2	0	0	(1,0)	0	0	0	(2,0)	0	0	0	(3,0)	0	0	0	(1,0)	0
0 n_3 0 $_2$	0	0	0	(1,0)	0	0	0	(2,0)	0	0	0	(3,0)	0	0	0	(4,0)
J	(0,1)	J	J	J	(0,1)	J	J	J	(1,1)	J	J	J	(1,1)	J	J	J
J n_1	J	(0,1)	J	J	J	(2,1)	J	J	J	(1,1)	J	J	J	(1,1)	J	J
J n_2	J	J	(0,1)	J	J	J	(2,1)	J	J	J	(3,1)	J	J	J	(1,1)	J
J n_3 J $_2$	J	J	J	(0,1)	J	J	J	(2,1)	J	J	J	(3,1)	J	J	J	(4,1)
0	(0,0)	0	0	0	(1,0)	0	0	0	(1,0)	0	0	0	(1,0)	0	0	0
2J n_1	2J	(0,2)	2J	2J	2J	(1,2)	2J	2J	2J	(1,2)	2J	2J	2J	(1,2)	2J	2J
2J n_2	2J	2J	(0,2)	2J	2J	2J	(1,2)	2J	2J	2J	(3,2	2J	2J	2J	(1,2)	2J
2J n_3 $_2$	2J	2J	2J	(0,2)	2J	2J	2J	(1,2)	2J	2J	2J	(3,2)	2J	2J	2J	(4,2)
J	(0,1)	J	J	J	(1,1)	J	J	J	(0,1)	J	J	J	(1,1)	J	J	J
J n_1	J	(0,1)	J	J	J	(1,1)	J	J	J	(2,1)	J	J	J	(1,1)	J	J
3J n_2	3J	3J	(0,3)	3J	3J	3J	(1,3)	3J	3J	3J	(2,3)	3J	3J	3J	(1,3)	3J
3J n_3 $_2$	3J	3J	3J	(0,3)	3J	3J	3J	(1,3)	3J	3J	3J	(2,3)	3J	3J	3J	(4,3)
J	(0,1)	J	J	J	(1,1)	J	J	J	(0,1)	J	J	J	(1,1)	J	J	J
J n_1	J	(0,1)	J	J	J	(1,1)	J	J	J	(2,1)	J	J	J	(1,1)	J	J
J n_2	J	J	(0,1)	J	J	J	(1,1)	J	J	J	(2,1)	J	J	J	(3,4)	J
4J n_3 n_4	4J	4J	4J	(0,4)	4J	4J	4J	(1,4)	4J	4J	4J	(2,4)	4J	4J	4J	(3,4)

(23)

Thus N^c exceeds the lower bound **(21)** by 18, two fewer than the 20 guaranteed by Theorem **10.**

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Resolution IV fractional factorial designs permit estimates of all main effects in the presence of two factor interactions which may not be estimable. A lower found on the number of runs required for a resolution IV design in the → next page →
$\prod_{i=1}^{k} s_i^{n_i} \times s_1^{n_1} \times s_2^{n_2} \times \dots \times s_k^{n_k}$ factorial is $N \geq s_k[\sum_{i=1}^{k}(s_i-1)n_i - (s_k-2)]$
where $s_k \geq s_{k-1}, s_{k-2}, \dots, s_1$. No series of designs are known which meet this

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bound except for the 2^n series and the trivial case where there are two factors. In this paper a method of construction is given which yields resolution IV designs near the theoretical (perhaps unattainable) lower bound. For the $2^m \times 3^n$ factorial, the designs exceed the lower bound by six if $n \geq 3$, by four if $n = 2$, and by three if $n = 1$. More generally for the $s_1^n \times s_2^n$, $s_1 < s_2$, the designs exceed the lower bound by $s_2(s_2-1)$ if $n \geq 3$, by $s_2-3s_2+2s_1$ if $n = 2$, and by $s_2(s_1-1)$ if $n = 1$. In general the designs never exceed the lower bound by $s_k(s_k-1)$.